

Hierarchical Spherical Model from a Geometric Point of View

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Abstract A continuous version of the hierarchical spherical model at dimension $d = 4$ is investigated. Two limit distributions of the block spin variable X^γ , normalized with exponents $\gamma = d + 2$ and $\gamma = d$ at and above the critical temperature, are established. These results are proven by solving certain evolution equations corresponding to the renormalization group (RG) transformation of the $O(N)$ hierarchical spin model of block size L^d in the limit $L \downarrow 1$ and $N \rightarrow \infty$. Starting far away from the stationary Gaussian fixed point the trajectories of these dynamical system pass through two different regimes with distinguishable crossover behavior. An interpretation of this trajectories is given by the geometric theory of functions which describe precisely the motion of the Lee–Yang zeroes. The large- N limit of RG transformation with L^d fixed equal to 2, at the criticality, has recently been investigated in both weak and strong (coupling) regimes by Watanabe (J. Stat. Phys. 115:1669–1713, 2004). Although our analysis deals only with $N = \infty$ case, it complements various aspects of that work.

Keywords Hierarchical model · Local potential approximation · Lee–Yang zeroes · Limit distributions · Geometric theory · Conformal mapping

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1 Introduction and Statement of Results

We continue the investigation starting in [2]. In the present work we give a geometric interpretation to certain trajectories of a first order partial differential equation related to the renormalization group transformation (RGT) of a d -dimensional hierarchical spherical model.

1.1 Motivation

The hierarchical $O(N)$ spin model, with $L^d = 2$ sites per block, has been recently studied by renormalization group in both weak and strong regimes by Watanabe [21]. Starting from the uniform “a priori” measure supported in the N -dimensional sphere of radius \sqrt{N} , the critical trajectory of the RGT has shown to converge to the Gaussian fixed point for sufficiently large N . To control such trajectory, which starts far away from the fixed point, the exactly solved $O(\infty)$ trajectory has been used together with two key ingredients: reflection positivity and the Lee–Yang property of single-site “a priori” measures. The former ingredient gives uniform convergence of $O(N)$ trajectories to $O(\infty)$ trajectories. The latter property has been previously employed by Kozitsky [15] to establish two central limit theorems. Watanabe’s analysis, based in his joint work with Hara and Hattori [11] on the critical trajectory for the hierarchical Ising model ($N = 1$), in contradistinction to Kozitsky’s, and most of the previous studies of this model, does not restrict the space of “a priori” measures to a neighborhood of the Gaussian fixed point and is able to deal with the borderline $d = 4$ case.

Although the analysis of the RGT with $L^d \geq 2$ fixed is expected to be simplified considerably in the $L \downarrow 1$ limit (see e.g. [9]), none of the above mentioned results can be carried to the limit as the two key ingredients do not hold if L^d is not an integer. In order to establish, in the local potential approximation ($L \downarrow 1$), a weak convergence of the hierarchical $O(N)$ Heisenberg equilibrium measure to the corresponding spherical equilibrium measure as $N \rightarrow \infty$ an entirely new method of analysis has to be developed from scratch.

In the present investigation we establish central limit theorems for the four-dimensional hierarchical spherical ($N = \infty$) model at and above the critical temperature. Our results are achieved in the local potential approximation that reduces the renormalization group equation to a nonlinear first order partial differential equation. A geometric function interpretation of the $O(\infty)$ trajectory is thus given with the help of an explicit solution obtained by the method of characteristics. It follows from our analysis that the Lee–Yang zeroes reach a limit distribution as the Gaussian fixed point approaches but their support moves away to infinity.

1.2 The Model

The hierarchical Heisenberg model on a finite box $\Lambda_K = \{0, 1, \dots, L^K - 1\}^d \subset \mathbb{Z}^d$ of size $n = L^{dK}$ is given by the $O(N)$ invariant equilibrium measure

$$dv_n^{(N)}(\mathbf{x}) = \frac{1}{Z_n^{(N)}} \exp\left\{\frac{1}{2}(\mathbf{x}, A\mathbf{x})_{\Omega_n}\right\} \prod_{j=1}^n d\sigma_0^{(N)}(x_j) \quad (1.1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ denotes an element of the configuration space $\Omega_n = \mathbb{R}^N \times \dots \times \mathbb{R}^N$; $A = J \otimes I$ the tensor product of the hierarchical coupling matrix J (whose quadratic form

$$\begin{aligned} (s, -Js)_{\Lambda_K} &= -(L-1) \sum_{k=1}^K L^{-2k} \sum_{r \in \Lambda_{K-k}} (B^k s)_r^2, \\ (Bs)_i &= \frac{1}{L^{d/2}} \sum_{j \in \{0, \dots, L-1\}^d} s_{Li+j}, \end{aligned} \tag{1.2}$$

coincides with Dyson’s hierarchical energy [7] when there are $L^d = 2$ sites per block¹ with the $N \times N$ identity matrix I ; $\sigma_0(x)$ the “a priori” uniform measure on the N -dimensional sphere $|x|^2 = \beta N$ of radius $\sqrt{\beta N}$ with β the inverse temperature.

1.3 Recursion Relations

The invariance of J under block transformation (1.2) allows to establish a recursion relation:

$$\sigma_k^{(N)}(x) = \frac{1}{C_k} e^{c_\gamma(L-1)|x|^2/2} \underbrace{\sigma_{k-1}^{(N)} * \dots * \sigma_{k-1}^{(N)}}_{L^d\text{-times}}(L^{\gamma/2}x) \tag{1.3}$$

on the space of single-site “a priori” measures in \mathbb{R}^N with initial data $\sigma_0^{(N)}(x)$. Here, $*$ denotes the convolution product

$$\rho * \eta(x) = \int_{\mathbb{R}^N} \rho(x - x') d\eta(x'),$$

C_k is chosen so that $\sigma_k^{(N)}$ is a probability measure and

$$c_\gamma = \begin{cases} 1 & \text{if } \gamma = d + 2, \\ L^{-2k} & \text{if } \gamma = d. \end{cases} \tag{1.4}$$

The “a priori” measure $\sigma_k^{(N)}$ at the step k is defined by integrating (1.1) over Ω_n with the value of k -th block spin fixed:

$$\int \delta((B^k \otimes I)\mathbf{y} - \mathbf{x}) d\nu_n^{(N)}(\mathbf{y}) = \frac{1}{Z_{L^{d(K-k)}}^{(N)}} \exp\left\{ \frac{1}{2}(\mathbf{x}, \mathbf{Ax})_{\Omega_{L^{d(K-k)}}} \right\} \prod_{j=1}^{L^{d(K-k)}} d\sigma_k^{(N)}(x_j)$$

is a marginal measure on $\Omega_{L^{d(K-k)}}$ that preserves the form (1.1).

In terms of their characteristic functions

$$\phi_k^{(N)}(z) = \int \exp(ix \cdot z) d\sigma_k^{(N)}(x), \tag{1.5}$$

(1.3) reads

$$\phi_k^{(N)}(z) = \frac{1}{N_k} \exp\left(\frac{-L+1}{2} c_\gamma \Delta \right) (\phi_{k-1}^{(N)}(L^{-\gamma/2}z))^{L^d} \tag{1.6}$$

¹The factor $L - 1$ is chosen so that the hierarchical Laplacean converges, as $L \downarrow 1$, to a continuum hierarchical Laplacean (see [2, 9]).

for $k \geq 1$ with

$$\phi_0^{(N)}(z) = \frac{\Gamma(N/2)}{(\sqrt{\beta N}|z|/2)^{N/2-1}} J_{N/2-1}(\sqrt{\beta N}|z|) := \varphi_0^{(N)}(|z|). \tag{1.7}$$

Here, $\exp(t\Delta)$ is the semi-group generated by the N -dimensional Laplacean operator $\Delta = \partial^2/\partial z_1^2 + \dots + \partial^2/\partial z_N^2$, N_k is chosen so that $\phi_k(0) = 1$ holds for all $k = 1, \dots, K$ and $J_\alpha(x)$ is the Bessel function of order α (see (20) in Chap. VII of [3] for an appropriate integral representation). Note that $\phi_k^{(N)}(z) = \varphi_k^{(N)}(r)$ depends only on $r = |z| = \sqrt{z \cdot z}$.

1.4 Thermodynamical Functions

The macroscopic behavior of the model is described by the limit distribution of the block variable

$$X_{n,N}^\gamma = \frac{1}{\sqrt{n^{\gamma/d}}} \sum_{j=1}^n x_j, \tag{1.8}$$

where γ is chosen in order the limit law to be attained. The characteristic function associated with the block variable $X_{n,N}^\gamma$ with $\gamma = d + 2$ is given by

$$\begin{aligned} \Phi_n^{(N)}(z) &= \int \exp\left(iL^{-K(d+2)/2} \left(\sum_{j=1}^n x_j\right) \cdot z\right) d\nu_n^{(N)}(\mathbf{x}) \\ &= \int \exp(ix \cdot z) d\sigma_K^{(N)}(x) = \varphi_K^{(N)}(|z|). \end{aligned}$$

As n goes to infinity, $X_{n,N}^\gamma$ converges in distribution to X_N^γ if $\varphi_K^{(N)}(r)$ converges at every point $r \geq 0$ to a function $\varphi^{(N)}(r)$ that is continuous at $r = 0$, by continuity theorem (see e.g. [7]). The convergence of $\nu^{(N)} = \lim_{n \rightarrow \infty} \nu_n^{(N)}$ to the equilibrium measure ν of the spherical model is more subtle and we analogously employ: X_N^γ is said to converge to X^γ in distribution if

$$\lim_{N \rightarrow \infty} (\varphi^{(N)}(\sqrt{Nr}))^{1/N} = \varphi(r)$$

exist for every point $r \geq 0$, is continuous at $r = 0$ and coincides with the corresponding characteristic function of the spherical model. The rescaling is seen to be necessary already at the initial function $\varphi_0^{(N)}(r)$ (see Proposition 2.1).

The statements about convergence are independent of the order of the limits of $n \rightarrow \infty$ and $N \rightarrow \infty$. This has been shown in [2] adapting a method employed by Kac and Thompson [13] for the hierarchical equilibrium measure (1.1) with $\gamma = d$, $L^d \geq 2$ an integer and β different from the critical inverse temperature $\beta_c = \beta_c(d, L)$ of the hierarchical spherical model. In [2], X^d is shown (see Theorem 1.1 and Remark 4.4) to be Gaussian with mean zero and variance $-1/\mu$ where $\mu = \mu(\beta)$ is implicitly defined by

$$\beta = \int \frac{1}{\lambda - \mu} d\varrho(\lambda) \tag{1.9}$$

where $\varrho(\lambda)$ is the density of eigenvalues (counted multiplicities) of the hierarchical Laplacean $-\Delta_H = \frac{L-1}{L^2-1}I - J$. Note J is not invariant under translation by a vector in \mathbb{Z}^d ,

property that is required for coupling matrices in [13]. Some statements about hierarchical spherical model hold also in the limit as $L \downarrow 1$, in which case (1.9) reads (see Sect. 3 of [2])²

$$1 - \frac{\beta}{4} = -2\mu \ln\left(1 - \frac{1}{2\mu}\right) \tag{1.10}$$

for $d = 4$. Central limit theorems are established in the present work directly from the $L \downarrow 1$ limit.

1.5 Local Potential Approximation

Let

$$U(t, z) = -\ln \phi_k^{(N)}(z) \tag{1.11}$$

be defined for $t = k \ln L$. As $k \rightarrow \infty$ together with $L \downarrow 1$ so that $k \ln L$ is kept fixed at a positive real number t , (1.4) converges to

$$c_\gamma(t) = \begin{cases} 1 & \text{if } \gamma = d + 2, \\ e^{-2t} & \text{if } \gamma = d \end{cases}$$

and we have

$$\begin{aligned} U_t &= \lim_{L \downarrow 1} \frac{U(t, z) - U(t - \ln L, z)}{\ln L} \\ &= \lim_{k \rightarrow \infty} \frac{k}{t} \left\{ -\ln \left[\frac{1}{N_k} \exp \left\{ -\frac{t}{2k} c_\gamma \Delta \right\} (\phi_{k-1}^{(N)}(e^{-\gamma t/2k} z))^{e^{td/k}} \right] + \ln \phi_{k-1}^{(N)}(z) \right\}. \end{aligned}$$

Consequently, (1.11) satisfies the initial value problem

$$U_t = -\frac{c_\gamma}{2} (\Delta U - |U_z|^2) + dU - \frac{\gamma}{2} z \cdot U_z + \frac{c_\gamma}{2} \Delta U(t, 0) \tag{1.12}$$

with

$$U(0, z) = -\ln \phi_0^{(N)}(z). \tag{1.13}$$

The last term in the right hand side ensures that $U(t, 0) = 0$ for all $t \geq 0$. Note that this property is satisfied by the initial condition because of the normalization $\int d\sigma_0^{(N)}(x) = \phi_0^{(N)}(0) = 1$.

We shall prove two limit theorems (Theorems 2.2 and 4.1) summarized as

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} U(t, \sqrt{N}z) = \begin{cases} |z|^2 & \text{if } \beta = \beta_c, \\ -|z|^2/2\mu & \text{if } \beta < \beta_c \end{cases}$$

uniformly in compact subsets of $\zeta \in \mathbb{C}$ with $\Re e(\zeta) = -|z|^2$. The first, when the sum (1.8) is normalized with abnormal exponent $\gamma/d = 1 + 2/d$, holds at the critical point

$$\beta = \beta_c(d) = \frac{2d}{d-2}, \tag{1.14}$$

$d \geq 4$. The second, for normal exponent $\gamma/d = 1$, holds for any $\beta < \beta_c(d)$ and $d > 2$. In both cases only the borderline $d = 4$ will be considered for brevity.

²By monotonicity, there exist a unique solution $\mu = \mu(\beta) < 0$ defined for $0 < \beta < 4$.

1.6 Conformal Mapping

Although continuity at $|z| = 0$ suffices for these limit theorems, the ‘‘characteristic function’’ $\lim_{N \rightarrow \infty} \exp(\frac{1}{N}U(t, \sqrt{N}z))$ is shown to be an analytic function that converges, as $t \rightarrow \infty$, to an entire function. In addition, thanks to an explicit solution of the initial value problem (1.12) and (1.13) at $N = \infty$, the whole trajectory can be described by the geometric function theory.

The initial value (1.13) is a function of $|z|^2$ and (1.12) preserves this property. So, we define

$$u(t, x) = \lim_{N \rightarrow \infty} \frac{1}{N}U(t, \sqrt{N}z) \tag{1.15}$$

for $x = -|z|^2$ and let, for each $t \geq 0$, the partial derivative $u_x(t, \zeta)$ of u be extended as an analytic function of $\zeta = x + iy$ with $y > 0$. We prove in Theorem 3.2 that $u_x(t, \zeta)$, $t \geq 0$, map the upper half-plane \mathbb{H} conformally into a decreasing family of open convex sets

$$u_x(t, \mathbb{H}) = \Omega_t \subset \Omega_0 = u_x(0, \mathbb{H})$$

contained in \mathbb{H} , and there is a one-to-one and onto relation between this family and the trajectory \mathcal{O} at the critical inverse temperature $\beta_c(4) = 4$ converging to the Gaussian fixed point. Analogous theorem holds for the trajectory corresponding to normal fluctuations.

The boundary of Ω_t is the union of a segment $I_\alpha := [-\alpha, 0]$ extending from a point $-\alpha = -\alpha(t) < 0$ up to the origin over the real line and a convex curve $q = h(t, p)$, $p \in I_\alpha$, with $h(t, -\alpha) = h(t, 0) = 0$. $h(t, I_\alpha) = \{h(t, p), p \in I_\alpha\}$ encodes all informations about \mathcal{O} since it corresponds to the image of a branching cut of $u_x(t, \zeta)$. The principal branch of $u_x(t, \zeta)$ belongs to the Pick class of analytical function and admits to be represented as

$$u_x(t, \zeta) = -1 + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - 1/2} \right) d\mu(t, \lambda) \tag{1.16}$$

where $d\mu = \rho d\lambda$ is absolutely continuous (with respect to Lebesgue) Borel measure. Although (1.16) is not a canonical representation,

$$\rho(t, \lambda) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \Im(u_x(t, \lambda + i\eta))$$

holds as well. Denoting by $\Sigma(t) = (-\infty, -d(t))$ the support of μ in (1.16), we have

$$\begin{aligned} -\alpha(t) &= u_x(t, -d(t)), \\ h(t, I_\alpha) &= \Im(u_x(t, \Sigma(t) + i0)). \end{aligned}$$

The support $\Sigma(t)$ of $\mu(t, \lambda)$ determines the location of the Lee–Yang zeroes as it can be seen by representing $\varphi_k^{(N)}(r)$ into an infinite canonical product (for $\varphi_0^{(N)}(r)$, see proof of Proposition 2.1). By (1.11) and (1.15), these zeroes are poles of $u_x(t, \zeta)$ that become dense over the semi-line $\Sigma(t)$ as $N \rightarrow \infty$. As t goes to ∞ , $\alpha(t) \rightarrow 3/2$, $d(t) \rightarrow \infty$ leading $\Sigma(t)$ to an empty set \emptyset as all Lee–Yang singularities are expelled to infinity. As a consequence, $u_x(t, \zeta) \rightarrow -1$ uniformly in each compact set of \mathbb{C} .

The motion of the Lee–Yang zeroes can be attained from the moments of their distribution [19]. The moments satisfy an infinite system of ordinary first-order differential equations which is reduced in [11, 21] to a finite system by Lee–Yang inequalities. The presence

of one-dimension unstable manifold makes this system very sensitive to truncation and no simplification occurs in the limit $N \rightarrow \infty$. This has to be contrasted with the simple geometric analysis in Sect. 3 from which the dynamics of Lee–Yang zeroes can be described globally.

1.7 Outline

In Sects. 2 and 4 we prove Theorems 2.2 and 4.1, which are Gaussian limit laws for the spherical model on the local potential approximation. Section 3 presents an interpretation of explicit solution of the associate nonlinear first order partial differential equation according to the geometric function theory. A conclusion with final remarks is given in Sect. 5.

2 Central Limit Theorem

2.1 The Radial Equation

The initial value (1.13) is a function of $|z|^2 = r^2$ and the spherical symmetry is preserved by the evolution equation (1.12). So, it suffices to take into account the radial component of $z \cdot \partial/\partial z$ and Δ , respectively given by $r\partial/\partial r$ and

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial}{\partial r} \right) = \frac{\partial^2}{\partial r^2} + (N - 1) \frac{1}{r} \frac{\partial}{\partial r}.$$

Defining

$$u^{(N)}(t, x) = \frac{1}{N} U(t, \sqrt{N}z) \tag{2.1}$$

for $x = -|z|^2$, the initial value problem (1.12) and (1.13) for $\gamma = d + 2$ reads

$$u_t^{(N)} = \frac{2}{N} x u_{xx}^{(N)} + u_x^{(N)} - 2x(u_x^{(N)})^2 - \gamma x u_x^{(N)} + du^{(N)} - u_x^{(N)}(t, 0) \tag{2.2}$$

with $u^{(N)}(0, x) = U(0, \sqrt{N}z)/N$. As $N \rightarrow \infty$, the initial function converges to a limit:

Proposition 2.1

$$\lim_{N \rightarrow \infty} u^{(N)}(0, x) = \int_0^x \frac{-\beta}{1 + \sqrt{1 + 4\beta x'}} dx' \equiv u_0(x) \tag{2.3}$$

and the convergence is uniform in any compact set of the slit plane $\mathbb{C} \setminus (-\infty, -1/4\beta]$.

Proposition 2.1 is proven in Sect. 3. Watanabe established (2.3) writing $u_0(x)$ as a continued fraction of Gauss (see Lemma 4.1 of [20]). Additional properties are obtained by taking into account that u'_0 is an analytic function of the Pick class $P_{I(\beta)}$ which can be continued across the interval $I(\beta) = (-1/4\beta, \infty)$.

2.2 Viscosity Limit Equation³

Taking $N \rightarrow \infty$ in (2.2) we are led to a first order partial differential equation for (1.15)

$$u_t = u_x - 2xu_x^2 - \gamma xu_x + du - u_x(t, 0) \tag{2.4}$$

which can be solved by the method of characteristics. To avoid dealing with a nonlinear equation we apply the Legendre transformation to (2.4). Let

$$w(t, p) = \max_{x \geq 0} (xp - u(t, x)) = \bar{x}p - u(t, \bar{x}) \tag{2.5}$$

be the Legendre transform of u with respect to x where $\bar{x} = \bar{x}(t, p)$ is attained at the value x for which

$$p = u_x(t, x) \tag{2.6}$$

has a solution for every $t \geq 0$ and p in a certain domain depending on t .

Assuming $w(t, p)$ is a continuously differentiable and uniformly convex function of p such that $\lim_{p \rightarrow \infty} w(t, p)/|p| = \infty$ holds for all $t \geq 0$, the original function $u(t, x)$ can be recovered by inverse Legendre transformation

$$u(t, x) = \max_{p \in \mathbb{R}} (xp - w(t, p)) = x\bar{p} - w(t, \bar{p}) \tag{2.7}$$

where $\bar{p} = \bar{p}(t, x)$ solves $x = w_p(t, p)$ for p . Note that, by differentiating (2.5) with respect to t and p together with (2.6), we have

$$\begin{aligned} w_t &= -u_t, \\ w_p &= \bar{x} + (p - u_x(t, \bar{x}))\bar{x}_p = \bar{x}. \end{aligned} \tag{2.8}$$

Hence, w_p solves (2.6) for x . We are going to show that $w_p(t, p)$ is a monotone increasing function of p for every $t \geq 0$ therefore, $w(t, p)$ is convex and a well defined Legendre transform of u which, by (2.7), is also uniformly convex. It follows by duality of the Legendre transformation that

$$\bar{p}(t, x) = u_x(t, x) \tag{2.9}$$

which, in view of the presence of $u_x(t, 0)$ in (2.4), gives

$$u(t, x) = \int_0^x \bar{p}(t, x') dx'. \tag{2.10}$$

Using $\gamma = d + 2$ together with (2.5) and (2.8), (2.4) becomes

$$w_t = -p + 2p(1 + p)w_p + dw + \bar{p}_0$$

where $\bar{p}_0 = \bar{p}_0(t)$ is implicitly defined by the equation $0 = w_p(t, p)$. Writing $v = w_p = \bar{x}$ we arrive, by differentiating both sides of the above equation with respect to p , at the following initial value problem

$$v_t - 2p(1 + p)v_p = -1 + (\gamma + 4p)v \tag{2.11}$$

³ $1/N$ plays the role of viscosity since it is in front of the Laplacean as in the hydrodynamic equation of incompressible fluid. Viscosity solution (or limit) also refers to a method for obtaining “weak solutions” of semilinear first order partial differential equations (see e.g. [8]).

with

$$v(0, p) = \frac{1}{2p} + \frac{\beta}{4p^2} \equiv v_0(p). \tag{2.12}$$

Note that $v(0, p) = \bar{x}(0, p)$ is the value x that solves (2.6) at $t = 0$:

$$p = u_x(0, x) = u'_0(x) = \frac{-\beta}{1 + \sqrt{1 + 4\beta x}} \tag{2.13}$$

by (2.8) and (2.3).

2.3 Main Result

Our main result of this section is as follows

Theorem 2.2 *Equations (2.11) and (2.12) with $d = 4$ ($\gamma = 6$) are solved by*

$$v(t, p) = \frac{1}{2p} + \frac{1}{p^2} - \frac{4 - \beta}{4p^2} e^{2t} - \frac{1 + p}{p^3} \ln(1 + p - pe^{2t}). \tag{2.14}$$

At $\beta = \beta_c = 4$, there is a unique solution $\bar{p} = \bar{p}(t, x)$ of

$$v(t, p) = x \tag{2.15}$$

holomorphic in a neighborhood of origin, that converges, as $t \rightarrow \infty$, to -1 in every compact set of \mathcal{C} . Together with (2.10) and (2.1), this implies convergence to the Gaussian equilibrium solution of (1.12):

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} U(t, \sqrt{N}z) = |z|^2$$

uniformly in compacts.

Remark 2.3 The use of Legendre transform in the renormalization group transformation for the $O(N)$ Heisenberg model in the large- N limit goes back to Shang–Keng Ma’s work (see [17] and references therein). It is also reminiscent of the method of Laplace (see eqs. (3.1.13)–(3.1.16) of [14]). In ref. [21], Watanabe solved the discrete flow equation (1.6) with $L^d = 2$ in the $N \rightarrow \infty$ limit and partial differential equation is employed only for the heat semigroup part in (1.6). Theorem 2.2 extends Watanabe’s result to the flow equation (1.12) at the $L \downarrow 1$ limit.

Remark 2.4 Theorem 2.2 treats the border case $d = 4$ but holds for any $d \geq 4$. The proof of the theorem can also be adapted to deal with the convergence to nontrivial equilibrium solutions of (1.12) at $\beta = \beta_c(d)$, given by (1.14), for $2 < d < 4$. For instance, we have

$$v(t, p) = \frac{1}{2p} + \frac{3}{2p^2} - \frac{6 - \beta}{4p^2} e^t - \frac{3\sqrt{(1+p)/p}}{4p^2} \ln \frac{(1 - e^{-t}\sqrt{(1+p)/p})(1 + \sqrt{(1+p)/p})}{(1 + e^{-t}\sqrt{(1+p)/p})(1 - \sqrt{(1+p)/p})}$$

for $d = 3$. At $\beta_c(3) = 6$ the solution $\bar{p} = \bar{p}(t, x)$ of (2.15) converges, as $t \rightarrow \infty$, to a function defined for $x \geq x_0 \simeq -0.15$ whose profile is shown in Fig. 1. In this case, the limit function $\lim_{N \rightarrow \infty} \exp(-U(t, \sqrt{N}z)/N)$ differs significantly from the Gaussian function for $|z|$ small.

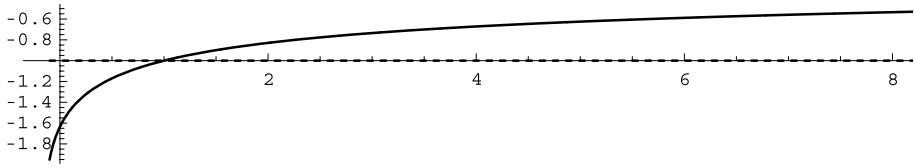


Fig. 1 The profile of $\lim_{t \rightarrow \infty} \bar{p}(t, x)$ for a trivial (dotted line) and nontrivial (solid line) fixed points

Proof Theorem 2.2 will be proven by solving (2.11) along the characteristics $p(t) = p(t; p_0)$ (see e.g. [8]). Writing $V(t) = v(t, p(t))$, (2.11) is reduced to a pair of ordinary differential equations

$$\begin{aligned} \dot{p} &= -2p(1 + p), \\ \dot{V} &= -1 + (6 + 4p)V \end{aligned} \tag{2.16}$$

satisfying initial conditions $p(0) = p_0$ and

$$V(0) = V_0 = v_0(p_0). \tag{2.17}$$

Integrating the first equation of (2.16)

$$\int_{p_0}^p \frac{dp'}{p'(1 + p')} = \int_{p_0}^p \left(\frac{1}{p'} - \frac{1}{1 + p'} \right) dp' = -2 \int_0^t dt'$$

gives

$$p(t) = \frac{p_0 e^{-2t}}{1 + p_0 - p_0 e^{-2t}}. \tag{2.18}$$

The second equation of (2.16) is a nonhomogeneous linear equation. The homogeneous equation $\dot{V} = (6 + 4p)V$ can be integrated:

$$\begin{aligned} V(t) &= V_0 \exp\left(6t + 4 \int_0^t p(s) ds\right) \\ &= V_0 e^{6t} (1 + p_0 - p_0 e^{-2t})^2. \end{aligned}$$

Using the variation of constants formula (see Theorem 3.1 of [1]), the solution to the second equation of (2.16) is given by

$$V(t) = e^{6t} (1 + p_0 - p_0 e^{-2t})^2 (V_0 - J_0) \tag{2.19}$$

with

$$J_0 = \int_0^t \frac{e^{-6s} ds}{(1 + p_0 - p_0 e^{-2s})^2},$$

by changing variable $\zeta = e^{-2s}$, given by

$$J_0 = \frac{1}{2p_0^3} \left[(1 + p_0)^2 \frac{1}{1 + p_0 - p_0 \zeta} + 2(1 + p_0) \ln(1 + p_0 - p_0 \zeta) + p_0 \zeta \right] \Big|_{\exp(-2t)}^1.$$

After some manipulations together with (2.17) and (2.12), this gives

$$V_0 - J_0 = \frac{\beta - 6}{4p_0^2} - \frac{1}{2p_0^3} + \frac{(1 + p_0)^2}{2p_0^3(1 + p_0 - p_0e^{-2t})} + \frac{e^{-2t}}{2p_0^2} + \frac{1 + p_0}{p_0^3} \ln(1 + p_0 - p_0e^{-2t}). \tag{2.20}$$

Equation (2.14) follows by plugging this result into (2.19) with p_0 as a function of t and p :

$$p_0(t, p) = \frac{pe^{2t}}{1 + p - pe^{2t}}$$

obtained by solving (2.18) for p_0 .

The Inverse Function Theorem We now solve (2.15) for p at the critical point $\beta = \beta_c(4) = 4$. By (2.14), it can be written as

$$xp^2 - \frac{p}{2} - 1 = -\frac{1 + p}{p} \ln(1 + p - pe^{2t}) \equiv g(t, p). \tag{2.21}$$

The first of two ingredients we need is

Lemma 2.5 For $p < (e^{2t} - 1)^{-1}$, g is a monotone increasing function of p diverging to $-\infty$ logarithmically as $p \rightarrow -\infty$ and satisfying $g(t, -1) = 0$ and $g(t, 0) = (e^{2t} - 1)$.

Proof of lemma Clearly, g is a well defined function of p for $1 + p - pe^{2t} = 1 - p(e^{2at} - 1) > 0$ with logarithmic divergence at $p = -\infty$. We have, by an explicit computation,

$$g_p(t, p) = \frac{e^{2t} - 1}{1 - p(e^{2t} - 1)} + \frac{1}{p^2} f(p(e^{2t} - 1))$$

where

$$f(w) = \ln(1 - w) + \frac{1}{1 - w} - 1 \equiv h(w) - 1. \tag{2.22}$$

If $f(w) \geq 0$ for all $w < 1$ then $g_p(t, p) > 0$ in the domain $p < (e^{2t} - 1)^{-1}$ and the monotonicity statement is proven. In fact, $h(0) = 1$ and

$$h'(w) = \frac{w}{(1 + w)^2}$$

implies that $w = 0$ is the absolute minimum of h proving an equivalent statement: $h(w) > 1$ for $w < 1$ different from 0. □

For $x \leq 0$, the quadratic polynomial

$$Q(x, p) := xp^2 - \frac{p}{2} - 1$$

in the left hand side of (2.21) is bounded from above by a linear function:

$$Q(x, p) \leq Q(0, p) = -\frac{p}{2} - 1,$$

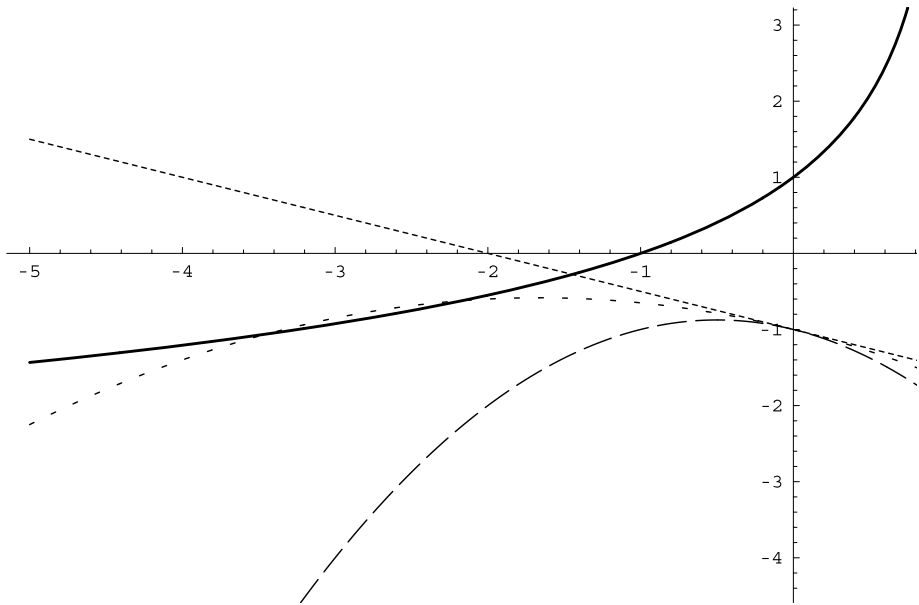


Fig. 2 Graphic of intersections of $g(t, p)$ (solid line) with $Q(x, p)$ for $x = 0$ (dotted line) and $x \neq 0$ (dashed lines)

and attains its maximum value $\frac{-1}{16x} - 1$ at $p_{\max} = \frac{1}{4x}$. Since $p_{\max} \rightarrow 0$ as $x \rightarrow -\infty$, there is a value $x_{\max} = x_{\max}(t)$ such that no real solutions of (2.15) exist for $x < x_{\max}$. On the other hand, as the graph of $g(t, p)$ intercepts the graph of $Q(x, p)$ in two points (one point) for any $0 > x > x_{\max}(t)$ ($x \geq 0$) and $t \geq 0$, there exist at least one real solution of (2.15) for $x \geq x_{\max}$ (see Fig. 2). We shall discard the solution associated with the second point of interception since it diverges at $x = 0$.

Now, let $t \geq 0$ and let x and p be real parts of numbers in \mathbb{C} : $z = x + iy$ and $\eta = p + iq$. Although the solution $\eta = \eta(t, z)$ of $z = v(t, \eta)$ is a multivalued function of z , only one branch, denoted by $\bar{\eta}(t, z)$, is regular at $z = 0$. Note that $\bar{\eta}(t, 0)$ exists for all $t \geq 0$ and is a real valued monotone increasing function of $t \geq 0$ satisfying $-2 \leq \bar{\eta}(t, 0) \leq -1$ as the graph of $g(t, p)$ always intercepts the straight line $Q(0, p) = -p/2 - 1$ at some negative point $p^*(t)$ within that range (see Fig. 2) and $p^*(t) = \bar{\eta}(t, 0)$ by definition.

It follows that $v(t, \eta)$ is holomorphic in $\Re(\eta) < 0$ with $v(t, \bar{\eta}(t, 0)) = 0$ and

$$v_{\eta}(t, \bar{\eta}(t, 0)) = \frac{1}{2\bar{\eta}^2} + \frac{1}{\bar{\eta}^4} \ln(1 - (e^{2t} - 1)\bar{\eta}) + \frac{1}{\bar{\eta}^2} \left(1 + \frac{1}{\bar{\eta}}\right) \frac{e^{2t} - 1}{1 - (e^{2t} - 1)\bar{\eta}} > 0 \quad (2.23)$$

and these are the assumptions of our second ingredient (see Theorem 9.4.1 of [12] for a proof).

Theorem 2.6 *Let $R > r > 0$ and $\bar{\eta} \in \mathbb{C}$ be such that $v(t, \eta)$ is holomorphic in $D_R(\bar{\eta}) = \{\eta \in \mathbb{C} : |\eta - \bar{\eta}| < R\}$, $v(t, \bar{\eta}) = 0$, $v_{\eta}(t, \bar{\eta}) > 0$ and $v(t, \eta) \neq 0$ for $0 < |\eta - \bar{\eta}| < r$. Then the contour integral*

$$\bar{\eta}(t, z) := \frac{1}{2\pi i} \int_C \eta \frac{v_{\eta}(t, \eta)}{v(t, \eta) - z} d\eta$$

where $\mathcal{C} = \{\eta \in \mathbb{C} : |\eta - \bar{\eta}| = \rho\}$ for some $\rho < r$, defines a holomorphic function in $\{z : |z| < m\}$ where

$$m = \min_{\theta} |v(t, \bar{\eta} + \rho e^{i\theta})|.$$

Moreover, $\eta = \bar{\eta}(t, z)$ is the unique solution of $z = v(t, \eta)$ regular at $z = 0$ in this domain.

For fixed t , let $R = R(t)$ be such that $D_R(\bar{\eta}(t, 0)) \subset \{\Re e(\eta) < 0\}$ and note that we can always take R large enough to include $\eta = -1$. Let $r < R$ be so that $v(t, \eta) \neq 0$ for $0 < |\eta - \bar{\eta}(t, 0)| < r$. This is always possible by continuity in view of (2.23). Finally we pick $\rho < r$ which gives the largest m . As t gets large, $\bar{\eta}(t, 0)$ approaches -1 and ρ may be chosen so that $m(t) = \min_{\theta} |v(t, \bar{\eta}(t, 0) + \rho e^{i\theta})|$ grows like t , namely, for ρ close to $1/2$. In the limit $t \rightarrow \infty$, $\bar{\eta}(t, z)$ becomes holomorphic in the entire complex plane.

To describe the asymptotic behavior of $\bar{\eta}(t, z)$ as $t \rightarrow \infty$, (2.15) can be written as

$$\begin{aligned} v(t, \bar{\eta}) &= \frac{1}{2\bar{\eta}^2} - \frac{\bar{\eta} + 1}{\bar{\eta}^3} \left\{ \frac{-\bar{\eta}}{2} + \ln(-\bar{\eta}) + 2t + \ln\left(1 - \frac{\bar{\eta} + 1}{\bar{\eta}} e^{-2t}\right) \right\} \\ &= \frac{1}{2} + 2t(\bar{\eta} + 1) + O(t(\bar{\eta} + 1)^2, (\bar{\eta} + 1)) = z \end{aligned}$$

which gives

$$\bar{\eta}(t, z) = -1 - \frac{1}{2t} \left(\frac{1}{2} - z \right) + R(t, z)$$

where, by Theorem 2.6, R is a regular function of z for $|z| < m(t)$ which goes to 0 faster than $1/t$, concluding the proof of Theorem 2.2. Note that

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} U(t, \sqrt{N}z) = \lim_{t \rightarrow \infty} u(t, x) = \int_0^x \lim_{t \rightarrow \infty} \bar{p}(t, x') dx' = -x = |z|^2$$

and $U_0(z) = |z|^2$ is an equilibrium solution of (1.12), for any number of components N . \square

Remark 2.7 Figure 3 shows the solution $v(t, p)$ of (2.11), for various t . For $t = 0$, $v(0, p) = v_0(p)$ is a monotone increasing (decreasing) function of $p \in (-4, 0)$ ($p \in (-\infty, -4)$) and its inverse $v_0^{-1}(x) = u'_0(x)$ is defined for $x \in (-1/16, \infty)$. For $t > 0$, there is a unique negative value $-l(t)$ (with $l(0) = 4$), given by $v_p(t, -l(t)) = 0$, such that $v(t, p)$ is monotone decreasing if $-\infty < p < -l(t)$ and monotone increasing if $-l(t) \leq p < 0$. The inverse function $v^{-1}(t, x) = u_x(t, x)$ has two branches but only the one with $v^{-1}(0, x) = u'_0(x)$ converges to -1 in any compact interval inside $(-d(t), \infty)$ with $-d(t) = v(t, -l(t)) < -1/16$ for all $t > 0$ and $d(t) \rightarrow \infty$ as $t \rightarrow \infty$.

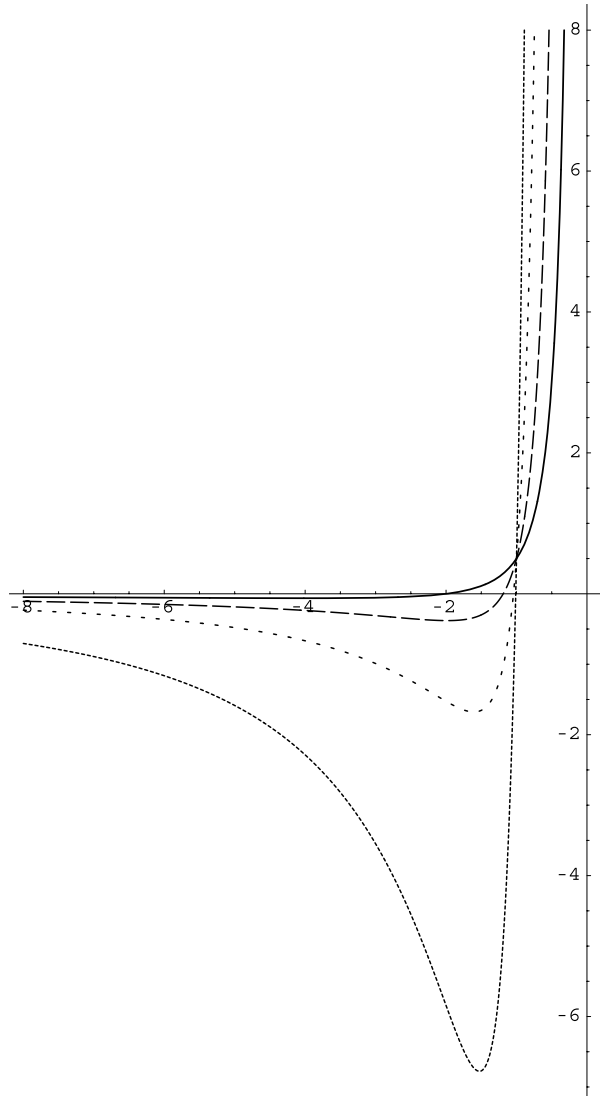
3 Geometry of the Scaling Flow

3.1 Critical Trajectory

The scaling flow $u(t, x)$, defined by (1.11) and (2.1), is the cummulant generating function of the block spin variable at scale t . The flow is determined by its partial derivative $u_x(t, x)$ (see (2.10)) and Theorem 2.2 exhibits a single trajectory, in the (viscosity) limit $N \rightarrow \infty$,

$$\mathcal{O}(u'_0 \rightarrow -1) = \{u_x(t, x), t > 0 : u_x(0, x) = u'_0(x), u_x(\infty, x) \equiv -1\},$$

Fig. 3 Profile of (2.14) for $t = 0$ (solid line), 10 (long dashes), 10^5 (short dashes) and 10^{20} (dots) and p in a neighborhood of $p = -1$



that starts at $t = 0$ from the initial function (2.13) and converges, as t goes to ∞ , to the stationary solution -1 , implicitly defined by (2.15) and (2.9). In this subsection we identify the class of functions where the flow is defined and give a geometric function theory description of this trajectory that establishes a one-to-one and onto relation between the orbit $\mathcal{O}(u'_0 \rightarrow -1)$ and the time dependent convex domains $\Omega(t) = u_x(t, \mathbb{H})$, $t \geq 0$, formed by images under u_x of the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$. Analytical and numerical techniques are combined in order the conformal equivalence between $\Omega(t)$ and \mathbb{H} to be explicitly verified for all t .

3.2 Analytic Continuation of Initial Value

Let us begin by extending Watanabe’s proof of Proposition 2.1 to the upper half-plane \mathbb{H} .

Proof of Proposition 2.1 Let $\phi_\nu(\xi) = \xi J_\nu(\xi) / J_{\nu-1}(\xi)$ be defined for $\nu \geq 1$ and $\xi \in \mathbb{C}$. The Bessel recursion relation

$$J_{\nu-1}(\xi) + J_{\nu+1}(\xi) = \frac{2\nu}{\xi} J_\nu(\xi)$$

generates a continued fraction of Gauss (see Chap. XVIII of [20]):

$$\phi_\nu(\xi) = \frac{2}{\nu} \frac{(\xi/2)^2}{1 - \frac{1}{2\nu} \phi_{\nu+1}(\xi)} = \frac{2}{\nu} \frac{(\xi/2)^2}{1 - \frac{1}{\nu(\nu+1)} \frac{1}{1 - \frac{1}{2\nu+2} \phi_{\nu+2}(\xi)}} \tag{3.1}$$

uniformly convergent over the domain

$$\frac{1}{\nu(\nu+1)} |\xi|^2 \leq 1, \tag{3.2}$$

by Worpitzky’s Theorem (see [20], p. 42).

Let $\vartheta_N(x) := U(0, \sqrt{Nz})/N$ with $x = -|z|^2$. Equation (1.13) together with (1.7) and the Bessel recursion relation $\nu J_\nu(\xi) - \xi J'_\nu(\xi) = \xi J_{\nu+1}(\xi)$, gives

$$\begin{aligned} x\vartheta'_N(x) &= \frac{1}{2N} \left\{ \frac{(N/2 - 1)J_{N/2-1}(i\sqrt{\beta x}N) - i\sqrt{\beta x}N J'_{N/2-1}(i\sqrt{\beta x}N)}{J_{N/2-1}(i\sqrt{\beta x}N)} \right\} \\ &= \frac{1}{2N} \phi_{N/2}(i\sqrt{\beta x}N). \end{aligned} \tag{3.3}$$

We take $\xi = i\sqrt{\beta x}N$ and $\nu = N/2$ in (3.1) and write

$$\frac{1}{2N} \phi_{N/2}(i\sqrt{\beta x}N) = \frac{-1}{2} \frac{a_0}{1 - \frac{a_1}{1 - \frac{a_2}{\ddots}}}$$

As N goes to infinity,

$$a_k = \frac{-\beta x}{(1 + \frac{2k}{N})(1 + \frac{2k+2}{N})}$$

converges to $-\beta x$ uniformly over the domain (3.2) for any integer $k \geq 0$ and, consequently, $x\vartheta'_N(x)$ converges over the same domain to a periodic continued fraction. We thus have

$$xu'_0(x) = \lim_{N \rightarrow \infty} x\vartheta'_N(x) = \frac{-1}{2} \frac{\beta x}{1 + \frac{\beta x}{1 + \frac{\beta x}{\ddots}}} = \frac{-\beta x}{1 + \sqrt{1 + 4\beta x}} \tag{3.4}$$

where the third equality is $-1/2$ times the solution ϕ of

$$\phi = \frac{\beta x}{1 + \phi}$$

that is positive for positive x . This yields (2.3) in view of the normalization $\vartheta_N(0) = 0$. Note that the limit holds for any x in the domain

$$|4\beta x| \leq 1 \tag{3.5}$$

of complex plane and this is sharp for the limit function u_0 since $(-\infty, -1/4\beta]$ is a branching cut of u'_0 .

Lee–Yang Zeroes Both functions $\vartheta'_N(x)$ and $u'_0(x)$ can be analytically continued to the upper half-plane and extended, by reflection, to the slit domain $\mathbb{C} \setminus (-\infty, -1/4\beta]$. As $\phi_v(\xi)$ is a meromorphic (even) function of ξ , it can be written as

$$\phi_v(\xi) = \xi^2 \sum_{n \geq 1} \frac{1}{\alpha_{n,v-1}^2 - \xi^2} \tag{3.6}$$

where $\alpha_{n,v}$, $n \geq 1$, are zeroes of the Bessel function J_v . So, the limit $N \rightarrow \infty$ of (3.3) together with the asymptotic behavior of the Bessel’s zeroes,

$$\alpha_{n,N/2-1} \sim (N-1)\frac{\pi}{4} + (2n-1)\frac{\pi}{2}$$

for n large, gives

$$u'_0(x) = \lim_{N \rightarrow \infty} \vartheta'_N(x) = \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{n \geq 1} \frac{-\beta}{\frac{\alpha_{n,N/2-1}^2}{N^2} + \beta x} = \frac{1}{2\pi} \int_{1/4}^{\infty} \frac{\beta}{-g(s) - \beta x} ds \tag{3.7}$$

for some positive function g satisfying $g(s) \sim s^2$ for large s . Note that $\{\alpha_{n,N/2-1}, n \geq 1\}$ are the Lee–Yang zeroes of the “a priori” initial measure (1.7) and, by (1.13) and (1.15), they become dense over an interval of real line.

Pick Class of Functions Let P denote the class of functions

$$f(\zeta) = u(\zeta) + iv(\zeta), \quad \zeta = x + iy,$$

analytic in the upper half-plane \mathbb{H} with positive imaginary part: $v(\zeta) \geq 0$ if $y > 0$ (see e.g. [6], Chap. II). The class of functions P forms a convex cone and is closed under composition:

- (1) $af_1 + bf_2 \in P$
- (2) $f_1 \circ f_2 \in P$

hold for any $a, b \geq 0$ and $f_1, f_2 \in P$.

A linear function $a + b\zeta$, $a \in \mathbb{R}$ and $b > 0$, and the function $-1/\zeta$ are clearly in P since both are one-to-one and onto maps of \mathbb{H} into itself. It thus follows by (3.6), together with the properties (1) and (2), that $\phi_v(\zeta)$ is in P and, in the topology of uniform convergence on compact subsets of \mathbb{H} , the sequence $(\vartheta'_N)_{N \geq 1}$ converges to ϑ'_∞ in P ([6], Sect. 4 in Chap. II). Note the following equality $\vartheta'_\infty(x) = u'_0(x)$ in the domain (3.5) and u'_0 is the composition of four Pick functions: $1 + \beta\zeta$, $\sqrt{\zeta}$, $(1 + \zeta)/\beta$ and $-1/\zeta$. This implies that $u'_0 \in P$ and equality between first and last expression in (3.4) holds with x replaced by $\zeta \in \mathbb{H}$, concluding the proof of Proposition 2.1. □

3.3 Integral Representation

A function $f(\zeta) = u(\zeta) + iv(\zeta)$ is in the Pick class if and only if has a unique canonical integral representation [6]

$$f(\zeta) = a\zeta + b + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda) \tag{3.8}$$

where $a = \lim_{y \rightarrow \infty} f(iy)/iy \geq 0$, $b = u(i)$ is real and μ is a positive Borel measure on \mathbb{R} such that $\int (\lambda^2 + 1)^{-1} d\mu(\lambda) < \infty$. In addition,

$$\mu((a, b)) + \frac{\mu(\{a\}) + \mu(\{b\})}{2} = \lim_{y \downarrow 0} \frac{1}{\pi} \int_a^b v(x + iy) dx \tag{3.9}$$

holds for any finite interval (a, b) and determines μ uniquely from f .

The initial condition $u'_0(x)$ of the flow $u_x(t, x)$ goes to 0 as x goes to infinity (in any direction of the complex plane). Consequently, a of its canonical representation vanishes. In addition, b can be identified with the second integral. So, if

$$f_0(\zeta) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \zeta} d\mu(\lambda) \tag{3.10}$$

is defined with $d\mu(\lambda) = \rho(\lambda)d\lambda$ an absolutely continuous measure w.r.t. the Lebesgue measure $d\lambda$:

$$\rho(\lambda) = \frac{1}{4\pi} \frac{\sqrt{4(-\lambda) - 1}}{(-\lambda)} \tag{3.11}$$

whose support is $-\infty < \lambda < -1/4$, then

$$u'_0(x) = \beta f_0(\beta x)$$

by (3.9). Note that $\int_{-\infty}^{\infty} \lambda^{-1} d\mu(\lambda) = -1/2$ agrees with $f_0(0)$, by an explicit integration. Equation (3.10) together with (3.7) leads to the following relation between (3.11) and the empirical density $\sqrt{g(s)}$ of Lee–Yang zeroes: $2\pi\rho(\lambda) = [(-g' \circ g^{-1})(\lambda)]^{-1}$.

3.4 Geometric Function Theory

By the Riemann mapping theorem (see e.g. [10]) if an open set Ω is topologically equivalent to \mathbb{H} (i.e. Ω and \mathbb{H} are homeomorphic) then Ω is also conformally equivalent to \mathbb{H} and there exist a biholomorphic (holomorphic one-to-one and onto) mapping f from Ω to \mathbb{H} . In some cases f can be made uniquely defined in Ω . The conformal equivalence of open sets provides qualitative informations on the trajectory $\mathcal{O}(u'_0 \rightarrow -1)$. As a function of the Pick class, u'_0 maps \mathbb{H} into itself but we can be more specific about the image of \mathbb{H} by u'_0 . From here on we fix β at the critical value $\beta_c(4) = 4$. We denote the upper semi-disc of radius r centered at x_0 by

$$\mathbb{S}_r(x_0) = \{\zeta = x + iy \in \mathbb{H} : (x - x_0)^2 + y^2 < r^2\}$$

and let \mathcal{S}_t be the class in \mathcal{P} indexed by $t \in \mathbb{R}_+$ satisfying

- (i) φ is a univalent function (one-to-one)
- (ii) $\varphi(\zeta_t) = \zeta_t$ for some complex number ζ_t
- (iii) $\varphi(1/2) = -1$
- (iv) $\varphi(\bar{\zeta}) = \bar{\varphi}(\zeta)$.

Proposition 3.1 u'_0 maps the upper half-plane \mathbb{H} conformally into the interior of the upper semi-disc of radius 2 centered at -2 :

$$u'_0(\mathbb{H}) = \Omega_0 = \mathbb{S}_2(-2)$$

and no other function in \mathcal{S}_0 maps \mathbb{H} into Ω_0 . Hence, there is a one-to-one and onto relation between Ω_0 and the initial function u'_0 of critical trajectory $\mathcal{O}(u'_0 \rightarrow -1)$ in the class \mathcal{S}_0 of functions with fixed point ζ_0 given by the complex root of $2x^3 - x - 2$.

Proof By (2.15) and (2.6), the inverse of u'_0 , given by

$$v_0(p) = \frac{p + 2}{2p^2}, \tag{3.12}$$

is the initial condition (2.12) of the linear evolution equation (2.11). Hence

$$\Omega_0 = \{\eta = p + iq \in \mathbb{H} : \Im(v_0(p + iq)) > 0\} \tag{3.13}$$

and this is equivalent, by (3.12), to the following inequalities

$$q(p^2 - q^2) - 2pq(p + 2) > 0, \quad q > 0$$

which can be written as the upper semi-disc $\mathbb{S}_2^+(-2)$: $(p + 2)^2 + q^2 < 4, q > 0$.

Since

$$v'_0(p) = -\frac{p + 4}{2p^3}$$

does not vanish or diverge for any p in Ω_0 but at edge points $p = -4$ and $p = 0$ in the closure $\overline{\Omega_0}$ of Ω_0 , we conclude by (3.12) that $u'_0(\mathbb{H}) = \Omega_0$ is a one-to-one and onto map. Note that $u''_0(x)$ vanishes at $x = v_0(p_\infty)$ with p_∞ such that $v'_0(p_\infty) = \infty$, in view of $u''_0 \circ v_0(p) = 1/v'_0(p)$, i.e., at infinity in every direction of the complex plane.

Now, suppose there exists another function $\varphi(x)$ in \mathcal{S}_0 such that $\varphi(\mathbb{H}) = \Omega_0$. Then, $\varphi^{-1} \circ u_x$ is a map from \mathbb{H} onto itself, belongs to the class P and leaves the points $1/2, \zeta_0$ and $\overline{\zeta_0}$ fixed.⁴ As a consequence of (3.8), the Pick functions that map \mathbb{H} onto \mathbb{H} are linear fraction transformations. Since the identity mapping is the only linear fraction transformation leaving three points fixed, we infer that $\varphi(x)$ and $u_x(x)$ are the same function. The complex root $\zeta_0 \simeq -0.582687 + 0.720119i$ of the fixed point equation

$$v_0(\zeta) = \frac{\zeta + 2}{2\zeta^2} = \zeta$$

is in Ω_0 , concluding the proof of Proposition 3.1. □

We now apply (3.13) to determine $\Omega_t = u_x(t, \mathbb{H})$ for $t > 0$. As $u_x(t, \zeta)$ solves $v(t, \eta) = \zeta$ for η , with v explicitly given by (2.14), the domain Ω_t can be easily plotted using ContourPlot or ImplicitPlot packages in *Mathematica*. Approximate expressions can be given for t around 0 and ∞ .

⁴The class of functions in P considered can be analytically continued across the real line by reflection (see condition (iv) of \mathcal{S}_t). If $\zeta_0 = x_0 + iy_0 \in \mathbb{H}$ is a fixed point of $f \in P$ then $\overline{\zeta_0} = x_0 - iy_0$ is a fixed point of its extension.

3.5 Domain Boundary

Each set Ω_t of the family for $t > 0$ is bounded by a simple convex closed curve which is piecewise analytic and defined by equation

$$\Im(v(t, \eta)) = 0, \quad \eta = p + iq \in \bar{\mathbb{H}} \tag{3.14}$$

where $v(t, \eta)$ is analytically continued to the closure $\bar{\mathbb{H}}$ of the half-plane \mathbb{H} . One has to be careful, however, in order to get the actual domain since $\Im(v(t, p + iq)) > 0$ may have more than one component. Figure 4 shows level curves of $\Im(v(t, p + iq))$.

For t small, Ω_t is a slight deformation of $\mathbb{S}_2(-2)$, by continuity:

$$\left(p + \frac{2(1+2t)}{1+4t} \right)^2 + q^2 < \frac{4(1+2t)^2}{(1+4t)^2}, \quad q > 0.$$

Whereas, for t very large, Ω_t approaches a folium (half-leaf) of Descartes:

$$\Im\left(2t \frac{1+p+iq}{(p+iq)^3}\right) \geq 0, \quad q > 0 \iff 2p(p^2+q^2) + 3p^2 - q^2 \leq 0, \quad q > 0.$$

We observe that the boundary of Ω_t is the union of two curves: a line segment $I_\alpha := [-\alpha, 0]$ extending from a point $-\alpha = -\alpha(t) < 0$ up to the origin over the real line and a convex curve $q = h(t, p)$ defined for $p \in I_\alpha$ with $h(t, -\alpha) = h(t, 0) = 0$. From the above, $\alpha(t)$ is a monotone decreasing function of t with $\alpha(0) = 4$ and $\lim_{t \rightarrow \infty} \alpha(t) = 3/2$ whereas

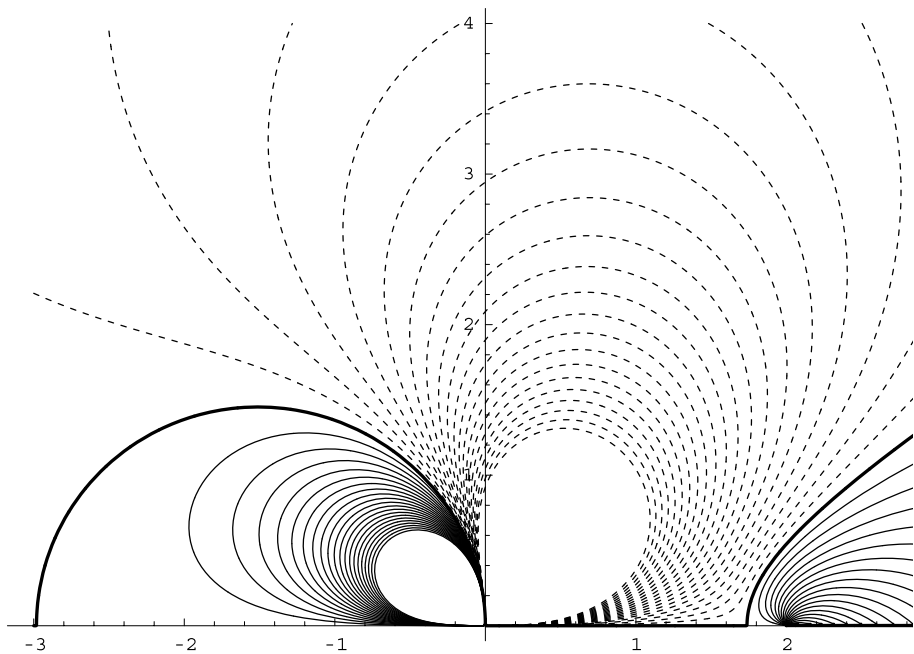


Fig. 4 $\Im(v(t, p + iq)) = c$ for $t = 0.2$ with c taking negative (dashed lines), positive (solid curves) and neutral (thick solid line) values

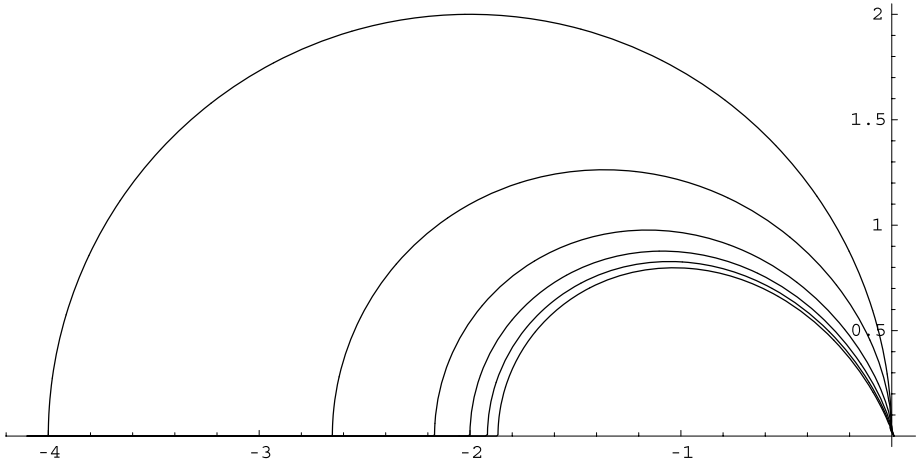


Fig. 5 Domain boundaries Ω_t for $t = n/4, n = 0, \dots, 9$

$h(t, p)$ is a semi-circular curve at $t = 0$: $h(0, p) = \sqrt{4 - (p + 2)^2}$ and approaches a limit (half-leaf) curve

$$h^*(p) = \lim_{t \rightarrow \infty} h(t, p) = \sqrt{\frac{3p^2 + 2p^3}{1 - 2p}}.$$

Figure 5 shows domain boundaries for various t . Note that $\Omega_t \subset \Omega_{t'}$ if $t' < t$ with strict inclusion along the convex arc.

The function $h(0, p)$ and the turning point $-\alpha(0) = -4$ are related to the density $\rho(\lambda)$ of the canonical representation (3.10) of $u'_0(x)$ and its support Σ_0 ,

$$\rho\left(\frac{1}{p}\right) = \frac{1}{4\pi} \sqrt{-4p - p^2} = \frac{1}{4\pi} h(0, p), \quad -4 \leq p \leq 0$$

by substituting $\lambda = 1/p$ in (3.11). Note that $\Sigma_0 = \Sigma(0) = (-\infty, -d(0))$ in this case is such that $d(0) = (4\alpha(0))^{-1} = 1/16$.

To determine the support $\Sigma(t) = (-\infty, -d(t))$ of the measure $\mu(t, d\lambda) = \rho(t, \lambda)d\lambda$ of the canonical representation of $u_x(t, x)$ we look at the negative value $-l(t)$ at which $v_p(t, -l(t)) = 0$. In the neighborhood of this point v is not univalent. Observe that $l(t)$ and the turning point $\alpha(t)$ coincide. Writing

$$v(t, \eta) = y(t, p, q) + iw(t, p, q), \quad \eta = p + iq, \tag{3.15}$$

by definition of α and Cauchy–Riemann equations, we have

$$0 = w_q(t, -\alpha(t), 0) = y_p(t, -\alpha(t), 0) = v_p(t, -\alpha(t))$$

which implies $\alpha(t) = l(t)$ by uniqueness. From Remark 2.7, we have $-d(t) = v(t, -l(t)) < -1/16$ for all $t > 0$ and

$$\begin{aligned} -d(t) &\sim v(t, -3/2) \\ &= \frac{1}{9} - \frac{4}{27} \ln\left(1 + \frac{3}{2}(e^{2t} - 1)\right) = \frac{-8}{27}t + O(1), \end{aligned}$$

for t large enough, implies that the support $\Sigma(t)$ of $\mu(t, \lambda)$ converges to an empty set: $\Sigma(t) = (-\infty, -d(t)) \rightarrow \emptyset$ as $t \rightarrow \infty$.

3.6 Riemann Surfaces

Contour plots of $v(t, \eta)$, $\eta \in \mathbb{C}$, for various t , show that $u_x(t, \zeta)$ is a multivalued function of $\zeta \in \mathbb{C}$. Already at $t = 0$, $u'_0(\zeta)$ has two Riemann surfaces connected by a branch cut along the segment $(-\infty, -1/16]$ across which the imaginary part of $u'_0(\zeta)$ change sign ($u'_0((-\infty, -1/16])$ is the semi-circular boundary of $\mathbb{S}_2(-2)$). The determination of $\sqrt{\cdot}$ is chosen such that $-1/(1 + \sqrt{1 + 16\zeta})$ is in P . For $t > 0$, $u_x(t, \zeta)$ has an even more elaborate Riemann surface with three sheets. The first is connected with the second sheet by a branch cut $(-\infty, -d(t)]$ while the latter is also connected to a third sheet by a branch cut $(0, d_1(t)]$ with $d_1(0) = 0$ and $d_1(t) \rightarrow \infty$ as $t \rightarrow \infty$, which does not concern us as it doesn't relate to the limit function -1 . The curves $u_x(t, (-\infty, -d(t)])$ and $u_x(t, (0, d_1(t)])$, which define together with the real line boundaries of two domains, intercept the real line perpendicularly at negative and positive values, respectively. Figure 6 shows these curves for various t . The region bounded by $u_x(t, (0, d_1(t)])$ inside the half-plane \mathbb{H} is denoted by Λ_t . Note that, opposed to Ω_t , Λ_t are open domains satisfying inclusions $\Lambda_t \subset \Lambda_{t'}$ if $t < t'$.

3.7 Flow in the Pick Class

It is very difficult to show directly from the flow equation that $u_x(t, \zeta)$ remains in the Pick class of functions for all $t > 0$ by general principles. However, for initial condition in P that belongs to the class \mathcal{S}_0 there is a simple property of the flow equation that explains why the Pick class P is preserved. Writing $v_0(\eta) = y_0(p, q) + iw_0(p, q)$ as a function of $\eta = p + iq \in \Omega_0 \cup \Omega_0^* \cup I_{\alpha(0)}$, with Ω^* the reflection of Ω about the real axis, if the imaginary part $w_0(p, q)$ is an odd function of q then the flow equation (2.11) preserves this property. Writing $v(t, \eta)$ as (3.15), we have

$$w(t, p, q) = -w(t, p, -q)$$

holds for all $t \geq 0$ and $\eta \in \Omega_t$. By continuity, it follows that $\Omega_t \subset \mathbb{H}$ and that $v(t, \eta)$ remains a one-to-one and onto map from Ω_t to \mathbb{H} and these imply that $u_x(t, \zeta)$ belongs to the class \mathcal{S}_t in P .

To establish uniqueness of the relationship between $\mathcal{O}(u'_0 \rightarrow -1)$ and the image domains $\{\Omega_t, t \geq 0\}$, we proceed as in $t = 0$ (see proof of Proposition 3.1). Supposing that

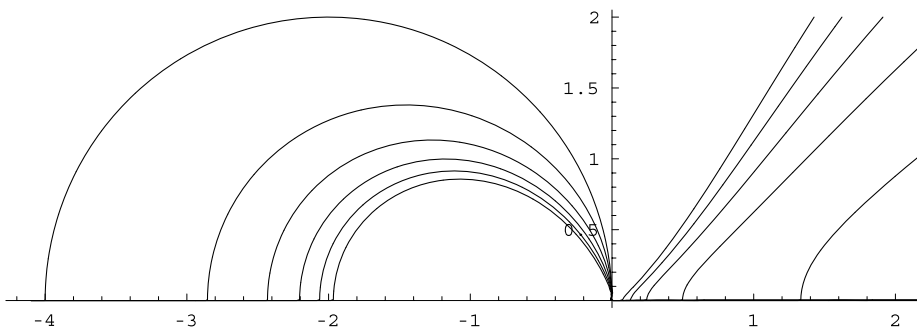


Fig. 6 Domain boundaries Ω_t and Λ_t for $t = n/4, n = 0, \dots, 9$

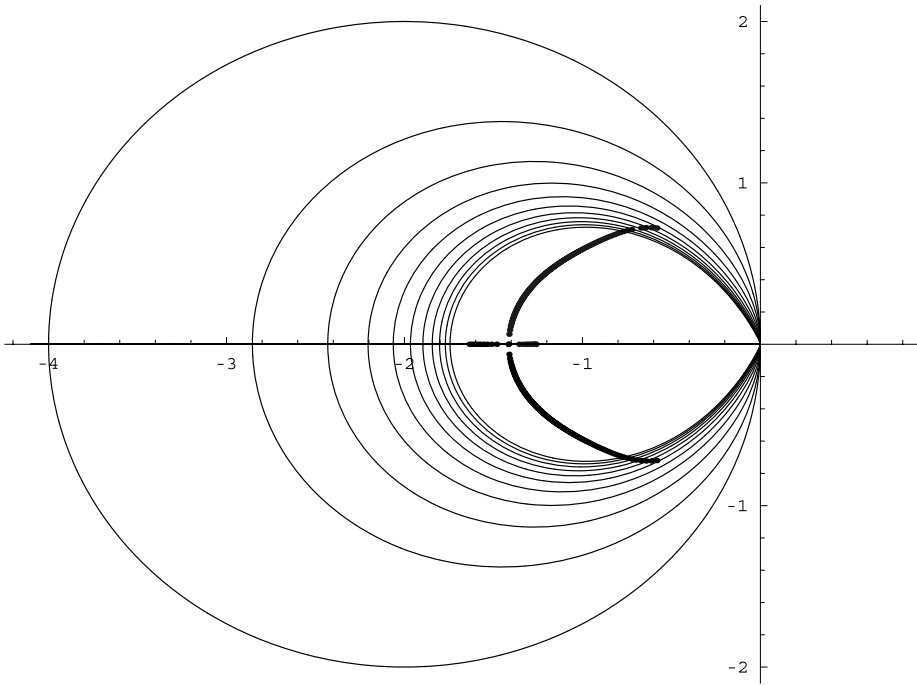


Fig. 7 Scaling evolution of fixed points and domain images of $u_x(t, \zeta)$

$\varphi(t, \zeta) \in \mathcal{S}_t$ is a different function satisfying $\varphi(t, \mathbb{H}) = \Omega_t$, for each t fixed $\varphi^{-1} \circ u_x(t, \zeta)$ maps \mathbb{H} into itself and leaves the point $1/2$, ζ_t and ζ_t^* fixed where $\zeta_t^* = \bar{\zeta}_t$ for $t < t^* \simeq 5.155075$ and for $t \geq t^*$ the last two fixed points become real numbers (see Fig. 7). Extending the functions in \mathcal{S}_t across the real line by reflection, $\varphi^{-1} \circ u_x(t, \zeta)$ is a linear fraction map with three fixed points which contradicts the hypothesis that $u_x(t, \zeta)$ and $\varphi(t, \zeta)$ are different. This holds for all t such that $\zeta_t^* \in \Omega_t \cup \Omega_t^* \cup I_{\alpha(t)}$. If this condition is not satisfied, we apply Schwarzian reflection [4] about the curve $h(p)$ in order to extend $u_x(t, \zeta)$ to the complex plane in such way that $u_x(t, \mathbb{H}) = \Omega_t$ and $u_x(t, -\mathbb{H}) = \mathbb{H} \setminus (\Omega_t \cup \Lambda_t)$ and this insures that ζ_t and ζ_t^* , which are now real values, remain fixed points of $u_x(t, \zeta)$ when $\zeta_t^* < -\alpha(t)$. The value t_{co} that $\zeta_{t_{co}}^* = -\alpha(t_{co})$ is called crossover scale from strong to weak (coupling) regime, term introduced in [11].

The canonical representation of $u_x(t, \zeta)$ is not suitable for describing the trajectory $\mathcal{O}(u'_0 \rightarrow -1)$. From the characteristic equations (2.16) of (2.11) one find that the point $(p, V) = (-1, 1/2)$ is a critical point for the two-dimensional dynamical system:

$$(\dot{p}, \dot{V}) = (-2p(1 + p), -1 + (6 + 4p)V) := (F_1(p, V), F_2(p, V)) \tag{3.16}$$

with $F_1(-1, 1/2) = F_2(-1, 1/2) = 0$. As $(-1, 1/2)$ is an invariant point we have $v(t, -1) = 1/2$ and, accordingly, $u_x(t, 1/2) = -1$. Instead of fixing b in the canonical representation (3.8) the value of f at $\zeta = i$, we write

$$u_x(t, \zeta) = -1 + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - 1/2} \right) d\mu(t, \lambda). \tag{3.17}$$

Note that, by hypothesis, $\int_{-\infty}^{\infty} [(\lambda - \zeta)(\lambda - 1/2)]^{-1} d\mu(t, \lambda) < \infty$ and as the support $\Sigma(t) = (-\infty, -d(t))$ of $\mu(t, \lambda)$ converges to \emptyset the integral in (3.17) converges to 0 uniformly in each compact set $O \in \mathbb{H}$.

The following summarizes our findings.

Proposition 3.2 $u_x(t, \zeta), t > 0$, map the upper half-plane \mathbb{H} conformally into a decreasing family of open convex sets Ω_t satisfying

$$\Omega_t = u_x(t, \mathbb{H}) \subset u'_0(\mathbb{H}) = \Omega_0$$

and no other function in \mathcal{S}_t maps \mathbb{H} into Ω_t . There is a one-to-one and onto relation between this family and the trajectory $\mathcal{O}(u'_0 \rightarrow -1)$ at the critical inverse temperature $\beta = \beta_c(4) = 4$. The geometric description together with the integral representation of $u_x(t, x)$ gives the distribution $d\mu(t, \lambda)$ of the Lee–Yang zeroes at the scale t . Ω_∞ is a nonempty set and a nontrivial limit distribution is attained but its support $\Sigma(t)$ is pushed away from the origin to infinity.

4 Normal Fluctuations

We turn our attention to normal fluctuations. The block variable (1.8) is now normalized with $\gamma = d$ and the system is above the critical temperature. The “a priori” measure $\sigma_K^{(N)}(x), L^{dK} = n$, that governs the law of (1.8), satisfies a recursive equation

$$\sigma_k^{(N)}(x) = \frac{1}{C_k} e^{L^{-2k}(L-1)|x|^2/2} \underbrace{\sigma_{k-1}^{(N)} * \dots * \sigma_{k-1}^{(N)}}_{L^d\text{-times}}(L^{d/2}x), \quad k \geq 1$$

which, in view of $\gamma = d$, has an explicitly k dependence in the exponential pre-factor (see (1.3)).

4.1 Initial Value Problem

Following the procedure described in Sect. 1, the initial value problem (1.12) and (1.13), for the logarithmic of its characteristic function $\phi_k^{(N)}(z)$ in the $L \downarrow 1$ limit, thus reads

$$U_t = -\frac{1}{2} e^{-2t} (\Delta U - |U_z|^2) + dU - \frac{\gamma}{2} z \cdot U_z + \frac{1}{2} e^{-2t} \Delta U(t, 0). \tag{4.1}$$

Note that, $L^{-2k} = \exp(-2k \ln L) \rightarrow \exp(-2t)$, as $k \rightarrow \infty$ together with $L \downarrow 1$ with $k \ln L = t$ fixed, and such function appears in front of the Laplacean in (1.6).

As $N \rightarrow \infty$, the radially symmetric solution of (4.1) scaled properly satisfies the modified initial value problem (see (1.15) for the definition of $u(t, x)$):

$$u_t = e^{-2t} u_x - 2x e^{-2t} u_x^2 - \gamma x u_x + du - e^{-2t} u_x(t, 0) \tag{4.2}$$

with $u(0, x) = u_0(x)$ given by (2.3).

We continue through (2.5)–(2.10). A similar Legendre transform applied to (4.2) leads to the initial value problem

$$v_t - 2p^2 e^{-2t} v_p = -e^{-2t} + (d + 4e^{-2t} p)v \tag{4.3}$$

with $v(0, p) = v_0(p)$ as given by (2.12). Note the cancellation of terms proportional to $p v_p$ because $\gamma = d$ in this case.

4.2 Main Result

The following result holds for any $d > 2$ but has been stated for $d = 4$, for simplicity.

Theorem 4.1 *Equations (4.3) and (2.12) with $d = 4$ are solved by*

$$v(t, p) = \frac{-e^{4t}}{p^2} \left(1 - \frac{\beta}{4} + \frac{1}{p} \ln(1 - p + pe^{-2t}) + e^{-2t} (\ln(1 - p + pe^{-2t}) - 1) - \frac{p}{2} e^{-4t} \right). \tag{4.4}$$

For every $\beta < \beta_c(4) = 4$ and $t \geq 0$, there is a unique solution $\bar{p} = \bar{p}(t, x)$ of

$$v(t, p) = x, \tag{4.5}$$

holomorphic in a neighborhood of the origin, that converges exponentially fast, as $t \rightarrow \infty$, to the solution of

$$1 - \frac{\beta}{4} = \frac{-1}{p} \ln(1 - p)$$

in every compact set of \mathbb{C} . This implies, together with the corresponding equations (2.10) and (2.1), convergence to a Gaussian equilibrium solution of (4.1) without terms proportional to e^{-2t} :

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} U(t, \sqrt{N}z) = \frac{-|z|^2}{2\mu(\beta)} \tag{4.6}$$

uniformly in compact subsets of $-|z|^2 \in \mathbb{C}$.

Proof As in the proof of Theorem 2.2, (4.3) will be solved along the characteristics $p(t; p_0)$. We refer to this proof for details. Writing $V(t) = v(t, p(t))$, we have

$$\begin{aligned} \dot{p} &= -2e^{-2t} p^2, \\ \dot{V} &= -e^{-2t} + (d + 4e^{-2t} p)V \end{aligned} \tag{4.7}$$

with initial conditions (2.17). Integrating the first of these equations gives

$$p(t) = \frac{p_0}{1 + p_0 - p_0 e^{-2t}}. \tag{4.8}$$

The homogeneous equation $\dot{V} = (d + 4e^{-2t} p)V$ can be integrated analogously as before

$$V(t) = V_0 e^{dt} (1 + p_0 - p_0 e^{-2t})^2.$$

Using the variation of constants formula, the solution to the second equation of (4.7) is given by

$$V(t) = e^{dt} (1 + p_0 - p_0 e^{-2t})^2 (V_0 - J_0) \tag{4.9}$$

with

$$J_0 = \frac{1}{2} \int_{\exp(-2t)}^1 \frac{\zeta^{d/2} d\zeta}{(1 + p_0 - p_0 \zeta)^2}.$$

At this point, notice that J_0 for $d = 4$ is exactly as in the proof of Theorem 2.2. Equations (2.17), (2.12) together with the integration of J_0 gives

$$V_0 - J_0 = (2.20).$$

The difference between the two cases is the exponential pre-factor e^{dt} of (4.9) and $p_0 = p_0(t, p)$ which is now obtained by solving (4.8) for p_0 :

$$p_0(t, p) = \frac{P}{1 - p + pe^{-2t}}. \tag{4.10}$$

As we shall see, these two differences are responsible for the converge of trajectories to different stationary solutions.

Equation (4.4) follows by plugging (2.20) into (4.9) with p_0 given by (4.10).

We now solve (2.15) for p at $\beta \neq \beta_c = 4$ which, by (2.14), can be written as

$$\left(xp^2 - \frac{p}{2}\right)e^{-4t} - e^{-2t} = -1 + \frac{\beta}{4} - \left(e^{-2t} - \frac{1}{p}\right)\ln(1 - p + pe^{-2t}) \equiv g_1(t, p). \tag{4.11}$$

Analogously to Lemma 2.5, we have

Lemma 4.2 *For any $p < (1 - e^{-2t})^{-1}$, g_1 is a monotone increasing function of p with $g_1(t, 0) = -e^{-2t} + \beta/4$ and diverges logarithmically to $-\infty$ as $p \rightarrow -\infty$.*

Proof of lemma g_1 is a monotone increasing function of p since

$$(g_1)_p(t, p) = \frac{e^{-2t}(1 - e^{-2t})}{1 - p(1 - e^{-2t})} + \frac{1}{p^2}f(p(1 - e^{-2t}))$$

with f given by (2.22) is a positive function for $p < (1 - e^{-2t})^{-1}$. Other statements follows as in the proof of Lemma 2.5. □

The quadratic polynomial $Q_1(x, p)$ in the left hand side of (4.11) tends to a linear function $Q_1(0, p) = -e^{-2t}(1 + pe^{-2t}/2)$ as $x \rightarrow 0$ with $Q_1(0, 0) = -e^{-2t}$ and $Q_1(0, -2e^{2t}) = 0$. From Lemma 4.2, the graph of g_1 always intercepts the graph of $Q_1(0, p)$ for all $\beta > 0$ and, as in the proof of Theorem 2.2, this implies the existence of a unique solution $\bar{p}(t, x)$ of (4.5) for every $t \geq 0$, holomorphic in a neighborhood $U(t)$ of the origin that becomes the entire complex plane $U(t) \rightarrow \mathbb{C}$ as $t \rightarrow \infty$. Details of the proof will be omitted since are similar to the corresponding statements in Theorem 2.2.

Asymptotic Expansion The asymptotic behavior of $\bar{p}(t, x)$ as $t \rightarrow \infty$ is given as follows. By (4.11), $\bar{p}(t, x)$ converges exponentially fast

$$\bar{p}(t, x) = \hat{p} \left(1 + \frac{4(\hat{p} + 2)}{\beta - 4\hat{p}(4 - \beta)} e^{-2t} + O(e^{-4t}) \right)$$

to a constant value \hat{p} which solves

$$1 - \frac{\beta}{4} = \frac{-1}{\hat{p}} \ln(1 - \hat{p}) \equiv h_1(\hat{p}). \tag{4.12}$$

Since h_1 is a monotone increasing function of $\hat{p} < 1$ with $h_1(0) = 1$ and $h_1(\hat{p}) \nearrow 0$ as $\hat{p} \rightarrow -\infty$, there is a unique strictly negative solution of (4.12) for all $0 \leq \beta < 4$. Comparing (4.12) with (1.10), we have

$$\hat{p} = \frac{1}{2\mu(\beta)}$$

which together with (1.11), (2.1) and (2.10), yields (4.6) and concludes the proof of Theorem 4.1. □

5 Conclusions and Final Remarks

In the present work, a continuous version of the hierarchical spherical model at dimension $d = 4$ has been investigated. The two main results are Theorems 2.2 and 4.1 on the limit distribution of the block spin variable X^γ normalized with exponent $\gamma = d + 2$ at the criticality and $\gamma = d$ above the critical temperature. To prove these results, certain evolution equations corresponding to the renormalization group transformation (1.6) in the limit $L \downarrow 1$ are solved explicitly at $N = \infty$. Starting far away from the stationary Gaussian fixed point the trajectories of these dynamical system pass through two different regimes with distinguishable crossover behavior. The large- N limit of the transformation (1.6) with L^d fixed equal to 2, at the criticality, has been investigated in both weak and strong (coupling) regimes by Watanabe [21]. We mention that our analysis using the $L \downarrow 1$ limit equation is considerably simpler and, consequently, has more details than Proposition 2.2 in [21].

Theorem 3.2 gives an interpretation for the above mentioned trajectories using the geometric function theory. The methods used enable us to describe the dynamics of the Lee–Yang zeroes along those trajectories. As $N \rightarrow \infty$, the Lee–Yang zeroes becomes dense over a semi-line and their measure, which depends on the scale parameter t , is shown to reach a limit for t large but the support of the limit measure is pushed away to infinity as the trajectories approach the Gaussian fixed point. The method also allow us to give the precise crossover scale $t_{co} > t^*$ from strong to weak regime defined as the value of t such that $\zeta_t^* = -\alpha(t)$ where ζ_t^* is a fixed point of the function (2.14) that solves (2.11) and $-\alpha(t)$ is a point of the boundary of image domain Ω_t .

There are, however, two major drawbacks in the $L \downarrow 1$ limit equation of the hierarchical $O(N)$ Heisenberg model with N finite. Firstly, reflection positivity cannot be used to prove uniform convergence of the $O(N)$ trajectories to $O(\infty)$ trajectories.

The other problem is related with the Lee–Yang property (for definition, see [15, 18]). A Borel measure ρ in \mathbb{R}^N possesses Lee–Yang property if its characteristic function $\phi(z) = \int d\rho(x) \exp(iz \cdot x)$ belongs to the Laguerre class \mathcal{L} of entire function of $\zeta = -|z|^2 \in \mathbb{C}$ which can be represented by

$$f(\zeta) = \exp(\lambda\zeta) \prod_{k=1}^{\infty} \left(1 + \frac{\zeta}{\alpha_k^2}\right) \tag{5.1}$$

with $\lambda \geq 0$ and $0 < \alpha_1 \leq \alpha_2 \leq \dots$ real numbers satisfying $\sum_{k=1}^{\infty} \alpha_k^{-2} < \infty$. Hence (see [11, 19, 21])

$$h(\zeta) = -\zeta(\ln f)'(N\zeta) = \sum_{j=1}^{\infty} (-1)^j v_{2j} \zeta^j \tag{5.2}$$

is holomorphic function of ζ in a neighborhood of the origin and Newman’s inequalities

$$0 \leq v_{2j} \leq (v_4)^{j/2} \tag{5.3}$$

holds for all $j \geq 2$. The scaling (5.2) is chosen so that $v_{2j} = O(1)$ in N for $j \geq 1$ if ρ is the uniform measure on the sphere of radius \sqrt{N} . Inequalities (5.3) holds in the limit $N \rightarrow \infty$ but in this case f cannot be represented by (5.1) as the zeroes $(\alpha_j)_{j \geq 1}$ become dense over the real line.

In order to include the $N \rightarrow \infty$ case the Lee–Yang property needs to be modified as follows. Let \mathcal{J} denote the set of probability measures ϱ on \mathbb{R} defined by

$$\phi_\varrho(\zeta) = \int e^{i\zeta s} d\varrho(s) := \left(\int e^{i\zeta \lambda \cdot x} d\rho(x) \right)^{-1/N} = e^{u(\zeta)}$$

for some $O(N)$ invariant measure ρ on \mathbb{R}^N possessing Lee–Yang property and $\lambda \in \mathbb{R}^N$. Notice that $u'(\zeta)$ belongs to the Pick class of functions satisfying (3.8) for some discrete measure μ . Let $\tilde{\mathcal{J}}$ be the closure of \mathcal{J} with respect to uniform convergence of $\phi_\varrho(\zeta)$ in compact subsets of \mathbb{H} (the natural topology of Pick functions).⁵ Then $\phi_\varrho(\zeta)$ is a infinitely divisible characteristic function [5] and $u(\zeta)$ is a completely monotonic function, $(-1)^n u^{(n)}(\zeta)$ on the semi-axis $\zeta \geq 0$ [6] and each is necessary and sufficient for the other. Using the integral representation (3.8) of $u'(\zeta)$ or Kolmogorov’s representation theorem for u , (5.3) can be verified. The present definition can be extended to accommodate the $L \downarrow 1$ limit since it has to be taken together with the thermodynamic limit and De Coninck’s results on Lee–Yang property and infinitely divisible characteristic function [5] can be applied. The solution $u(t, x)$ of (2.4) studied in the present paper defines, in this way, a one parameter family of characteristic functions $\exp(u(t, x))$ of probability measures in $\tilde{\mathcal{J}}$ that converges, at the critical inverse temperature, to $\exp(-x)$.

Now, let

$$f_k = T f_{k-1}, \quad k = 1, 2, \dots$$

where $T : \mathcal{E} \rightarrow \mathcal{E}$ is the operator defined by recursion relation (1.6) with $f(\zeta) = \varphi(|z|) = \phi(z)$, $\zeta = -|z|^2$, be a sequence in the space of entire functions \mathcal{E} starting from $f_0(\beta\zeta)$ with f_0 in the Laguerre’s class \mathcal{L} . It has been proven in Theorem 1.2 of [16] that, for every $k \in \mathbb{N}$ and $0 \leq \beta \leq (L^{\nu-d} - 1)/\lambda$,

$$f_k \in \mathcal{L} \cap \mathcal{A}_\lambda$$

where \mathcal{A}_a denotes the Fréchet space of functions $f \in \mathcal{E}$ such that

$$\|f\|_b := \sup_{k \in \mathbb{N}} \frac{1}{b^k} \left| \frac{d^k f}{d\zeta^k}(0) \right|$$

is finite for all $b > a$ and λ is the type of f_0 . This together with (5.3) can be used to establish the existence of a critical inverse temperature β_c such that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges to $\exp(\zeta)$ uniformly in compact subsets of \mathbb{C} . The Pick class of functions is the natural candidate for replacing Laguerre’s class in the local potential approximation of (1.6). The present work is the first attempt in this direction for $N = \infty$.

⁵We do not assume the condition $|\phi_\varrho(\zeta)| \leq C \exp(c|\zeta|)$ in the closure as it is required in [18] for the set of measures whose characteristic function is given by (5.1).

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